

Ergodicity for the $GI/G/1$ -type Markov Chain*

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Abstract

Ergodicity is a fundamental issue for a stochastic process. In this paper, we refine results on ergodicity for a general type of Markov chain to a specific type or the $GI/G/1$ -type Markov chain, which has many interesting and important applications in various areas. It is of interest to obtain conditions in terms of system parameters or the given information about the process, under which the chain has various ergodic properties. Specifically, we provide necessary and sufficient conditions for geometric, strong and polynomial ergodicity, respectively.

Keywords: Ergodicity, geometric ergodicity, strong ergodicity, polynomial ergodicity, tail asymptotics, geometric decay, light tailed, heavy tailed, queueing system, $GI/G/1$ -type.

1 Introduction

Ergodicity is a fundamental issue in the study of a stochastic process. There are many references in this area, among which closely related to our study include books by Anderson [1], Meyn and

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Tweedie [21], Chen [4], and references therein. Results in these books are usually presented for a general type of stochastic process, or Markov chain, while our focus is on a specific type of Markov chain; that is, the $GI/G/1$ -type Markov chain. We apply standard methods used for a general process, combined with various techniques in dealing with block-structured matrices, to characterize ergodic properties in terms of system parameters, which is an extension of the existing research.

The $GI/G/1$ -type Markov chain is a very important type of block-structured stochastic process with many applications in queueing theory; for example, see Grassmann and Heyman [7], Zhao, Li and Braun [23, 25], and Zhao [24]. We refine literature results on general Markov chains to obtain ergodicity conditions for this specific type of Markov chain. In this paper, we focus on the study of ergodicity of this type of Markov chain. For the $GI/G/1$ -type Markov chain, the ordinary ergodicity has been well studied. Necessary and sufficient conditions have been reported in the literature; for example, see Asmussen [2], Zhao, Li and Braun [23, 25], and Zhao [24]. This will not be discussed again in the current paper. Instead, we will consider three other types of ergodicity: geometric, strong (or uniform) and polynomial. Related results were reported in Spieksma and Tweedie [22].

Related to our research, Højgaard and Møller [8], using the coupling method and stopped random walks, provided a sufficient condition for geometric and polynomial ergodicity, respectively, for the $GI/G/1$ -type Markov chain. Hou and Liu [10] derived a necessary and sufficient condition for polynomial ergodicity for the $M/G/1$ queue by analyzing the generating function of the first return probability, and extended their study to the $M/G/1$ -type Markov chain in Liu and Hou [18]. Jarner and Tweedie [13] proved that for random-walk-type Markov chains, the geometric (light) and polynomial tail asymptotics in the stationary probability distribution are necessary for the geometric ergodicity and polynomial ergodicity, respectively.

The main contributions in this paper include necessary and sufficient conditions (Theorem 3.1, Theorem 4.1, Theorem 5.2 and Theorem 5.3) for each of these three types of ergodicity, given in terms of system parameters, or the given information about the $GI/G/1$ -type Markov chain.

Ergodicity and tail asymptotics of the stationary probability distribution are usually studied separately because of the obvious distinction between these two concepts. For a stochastic process, ergodicity deals with conditions under which the marginal distribution at time t converges to its limiting distribution in various speeds as t goes to infinity, while the tail asymptotic of the stationary (limiting) probability distribution is concerned with the speed to zero of the tail probabilities. It is interesting to observe that the same necessary and sufficient condition for both geometric ergodicity and a geometric decay in the stationary probability distribution immediately allows us to draw the conclusion of ergodicity on a number of important models for which the tail asymptotics have been known, and vice versa. The equivalence also opens a new door for us to take advantage of possibly using newly developed approaches and known results in studying ergodicity to study tail asymptotics of the stationary probability distribution. For a symmetric Metropolis-Hastings algorithm, Mengersen and Tweedie [20] and Jarner and Hansen [12] proved that these two concepts are actually equivalent. It is our goal to provide necessary and sufficient conditions for ergodicity for the $GI/G/1$ -type model in this paper.

The rest of the paper is organized into four sections. In Section 2, the $GI/G/1$ -type Markov

chain is reviewed, and a spectral property, Lemma 2.3, is obtained that plays a key role in proving the main result for geometric ergodicity. Sections 3–5 study geometric, strong and polynomial ergodicity, respectively.

2 The $GI/G/1$ -type Markov chain

Consider a discrete time irreducible aperiodic Markov chain, whose transition probability matrix is given by

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} & \cdots \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix} = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ B_{-1} & A_0 & A_1 & A_2 & \cdots \\ B_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ B_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}, \quad (2.1)$$

where A_i and B_i for $i = 0, \pm 1, \pm 2, \dots$, are matrices of size $m \times m$. The state space for the Markov chain P can be expressed by $S = \cup_{i=0}^{\infty} L_i$, where $L_i = \{(i, j); j = 1, 2, \dots, m\}$ for $i \geq 0$. In a state (i, j) , i is referred to as a *level* and j as a *phase*. We also write $L_{\leq i} = \cup_{k=0}^i L_k$.

Remark 2.1 *In fact, we can more generally assume that B_0 is a matrix of size $m_0 \times m_0$ with $m_0 \neq m$ for the results obtained in this paper.*

Along the same line as in [23], we define the R -measures $R_{i,j}$ for $i < j$ and the G -measures $G_{i,j}$ for $i > j$ for the $GI/G/1$ -type Markov chain. $R_{i,j}$ is a matrix of size $m \times m$ whose (r, s) th entry is the expected number of visits to state (j, s) before hitting any state in $L_{\leq (j-1)}$, given that the process starts in state (i, r) . $G_{i,j}$ is a matrix of size $m \times m$ whose (r, s) th entry is the probability of hitting state (j, s) when the process enters $L_{\leq (i-1)}$ for the first time, given that the process starts in state (i, r) . We refer to the matrices $R_{i,j}$ and $G_{i,j}$ as the matrices of the expected number of visits to higher levels before returning to lower levels and the matrices of the first passage probabilities to lower levels, respectively. From [23], we can write $R_{n-i} = R_{i,n}$ and $G_{n-i} = G_{n,i}$ for $i > 0$ due to the property of repeating rows. If the $GI/G/1$ -type Markov chain is positive recurrent, then the stationary distribution $\{\pi_k\}$ can be expressed in terms of the R -measures (see [7]):

$$\pi_n = \pi_0 R_{0,n} + \sum_{k=1}^{n-1} \pi_k R_{n-k}, \quad n \geq 1. \quad (2.2)$$

Define the generating functions for the stationary distribution $\{\pi_k\}$, the matrix sequences $\{R_{0,k}\}$ and $\{R_k\}$, respectively, as $\pi^*(z) = \sum_{k=0}^{\infty} z^k \pi_k$, $R_0^*(z) = \sum_{k=1}^{\infty} z^k R_{0,k}$ and $R^*(z) = \sum_{k=1}^{\infty} z^k R_k$. Then, we have that

$$\pi^*(z)[I - R^*(z)] = \pi_0 R_0^*(z). \quad (2.3)$$

Throughout this paper, we assume that the phase process $A = \sum_{k=-\infty}^{\infty} A_k$ is irreducible. Therefore, when A is stochastic in addition, there is a unique invariant probability vector μ of A , that is $\mu = \mu A$.

Definition 2.1 For a sequence $\{c_k\}$ of nonnegative scalars, it is called *light-tailed* if

$$\sum_{k=1}^{\infty} c_k e^{\varepsilon k} < +\infty$$

for some $\varepsilon > 0$. For a sequence $\{C_k\}$ of nonnegative matrices of size $m \times n$, it is called *light-tailed* if for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, the sequences $\{C_k(i, j)\}$ of nonnegative scalars are light-tailed, where $C_k(i, j)$ is the (i, j) th entry of C_k .

Let $A_+^*(z; r, s)$ and $B_+^*(z; r, s)$ be the (r, s) th entry of the matrix generating function $A_+^*(z) = \sum_{k=1}^{\infty} A_k z^k$ and $B_+^*(z) = \sum_{k=1}^{\infty} B_k z^k$, respectively. Denote by $\phi_{A+}(r, s)$ and $\phi_{B+}(r, s)$ the convergence radii of $A(z; r, s)$ and $B(z; r, s)$, respectively. Let

$$\phi_{A+} = \min_{r,s} \phi_{A+}(r, s)$$

and

$$\phi_{B+} = \min_{r,s} \phi_{B+}(r, s).$$

In Li and Zhao [15, 16], they proved the following.

Lemma 2.1 When A is stochastic, for the $GI/G/1$ -type Markov chain, the sequence of the stationary probability vectors $\{\pi_k\}_{k \geq 1}$ is light-tailed if and only if $\min(\phi_{A+}, \phi_{B+}) > 1$, or both $\{A_k\}_{k \geq 1}$ and $\{B_k\}_{k \geq 1}$ are light-tailed.

Define $A^*(z) = \sum_{k=-\infty}^{\infty} z^k A_k$. Then, according to Li and Zhao [16], we have the following spectral property.

Lemma 2.2 Suppose that the $GI/G/1$ -type Markov chain defined in (2.1) is irreducible, aperiodic and positive recurrent. If $\Omega = \{1 < |z| < \phi_{A+} : \det(I - A^*(z)) = 0\}$ is not empty, then there must exist a positive $z_0 \in \Omega$ such that $z_0 \leq |z|$ for $z \in \Omega$.

Define $\eta = z_0$ for $\Omega \neq \emptyset$; $\eta = \infty$ for $\Omega = \emptyset$. Let $\chi(z)$ for $z > 0$ be the largest eigenvalue of $A^*(z)$. We have the following spectral property.

Lemma 2.3 Assume that A is stochastic and $\phi_{A+} > 1$ for the irreducible aperiodic positive recurrent $GI/G/1$ -type Markov chain defined in (2.1). Then $\chi(z) < 1$ for any $1 < z < \min(\phi_{A+}, \eta)$, and there exists a Perron-Frobenius eigenvector $\mathbf{Y}(z)$ with $\mathbf{Y}(z) \geq \mathbf{e}^t$ such that

$$A^*(z)\mathbf{Y}(z) = \chi(z)\mathbf{Y}(z), \tag{2.4}$$

where \mathbf{e} is a row vector of ones and \mathbf{e}^t is the transpose of \mathbf{e} .

Proof: Without loss of generality, assume that $\eta < \phi_{A+}$. Since $\chi(1) = \chi(\eta) = 1$, it follows from the continuity of $\chi(\cdot)$ and the definition of η that we only need to show that $\chi'(1) < 0$.

Differentiating both sides of equation (2.4) at $z = 1$ gives

$$A^{*'}(1)\chi(1) + A^*(1)\mathbf{Y}'(1) = \chi'(1)\mathbf{Y}(1) + \chi(1)\mathbf{Y}'(1).$$

Note that because $A^*(1) = A$, $A^{*'}(1) = \sum_{k=-\infty}^{\infty} kA_k$ and $\mathbf{Y}(1) = \mathbf{e}^t$, the lemma follows from the ergodicity condition

$$\chi'(1) = \boldsymbol{\mu} \sum_{k=-\infty}^{\infty} kA_k \mathbf{e}^t < 0.$$

□

3 Geometric ergodicity

In this section, we present a necessary and sufficient condition for geometric ergodicity, which is the same condition for a geometric tail in the stationary probability distribution.

Let $P^n((i, r), (j, s))$ be the n -step transition probability for the irreducible and aperiodic Markov chain of $GI/G/1$ type given in (2.1). P is called geometrically ergodic if there exists a rate $\rho < 1$ such that

$$|P^n((i, r), (j, s)) - \pi_{j,s}| \leq M_{(i,r)(j,s)} \rho^n, \quad \text{for all } n \geq 0 \text{ and for all } i, j, r, s,$$

where $\pi_{j,s}$ is the stationary probability in level j and phase s and $M_{(i,r)(j,s)} < \infty$. It follows from Theorem 4.31 in Chen [3] that geometric ergodicity is equivalent to the following condition.

Condition 3.1 *There exist a finite set $H \neq \emptyset$, a constant $\lambda < 1$ and a finite function $V \geq 1$ defined on the state space of the Markov chain P , such that*

$$\begin{cases} PV(i, r) \leq \lambda V(i, r), & \text{for } (i, r) \notin H, \\ PV(i, r) < \infty, & \text{for } (i, r) \in H. \end{cases} \quad (3.5)$$

See also Anderson [1] or Meyn and Tweedie [21].

When A is stochastic, the $GI/G/1$ -type Markov chain is a special case of the random-walk-type Markov chain in Jarner and Tweedie [13], so it follows from Theorem 2.2 of [13] that the condition in Lemma 2.1 or $\min(\phi_{A+}, \phi_{B+}) > 1$ is necessary for P to be geometrically ergodic. To show the equivalence between the geometric ergodicity and geometric stationary tail as stated in Lemma 2.1, we only need to show that under the assumption $\min(\phi_{A+}, \phi_{B+}) > 1$, (3.5) holds.

Theorem 3.1 *Assume that A is stochastic and $\min(\phi_{A+}, \phi_{B+}) > 1$ for the irreducible aperiodic positive recurrent $GI/G/1$ -type Markov chain defined in (2.1). (3.5) holds for $V(i, r) = z^i y_r$ for any $1 < z < \min(\phi_{A+}, \phi_{B+}, \eta)$, where $\mathbf{Y} := \mathbf{Y}(z) = (y_1, y_2, \dots, y_m)^t$ is the right eigenvector of $A^*(z)$ corresponding to the eigenvalue $\chi(z)$ given in Lemma 2.3.*

Proof: For any fixed $1 < z < \min(\phi_{A+}, \phi_{B+}, \eta)$, $\delta := 1 - \chi(z) > 0$ by Lemma 2.3. Let N be large enough such that $\alpha z^{-N} \leq \delta/2$ or $N \geq (\log 2\alpha - \log \delta)/\log z$, where $\alpha = \max_{1 \leq r \leq m} y_r \geq 1$.

Set $H = L_{\leq N}$, a finite set. Then, for $(i, r) \notin H$ or $i > N$, we have

$$\begin{aligned}
P\mathbf{v}(i) &:= \sum_{k=0}^{\infty} P_{i,k} \mathbf{v}_k = B_{-i} \mathbf{v}_0 + A_{-i+1} \mathbf{v}_1 + A_{-i+2} \mathbf{v}_2 + A_{-i+3} \mathbf{v}_3 + \cdots \\
&= (B_{-i} z^{-i} + A_{-i+1} z^{-i+1} + A_{-i+2} z^{-i+2} + A_{-i+3} z^{-i+3} + \cdots) z^i \mathbf{Y} \\
&\leq z^i (B_{-i} z^{-i} \mathbf{Y} + A^*(z) \mathbf{Y}) \leq z^i (\alpha z^{-N} \mathbf{e}^t + A^*(z) \mathbf{Y}) \\
&= z^i (\alpha z^{-N} \mathbf{e}^t + (1 - \delta) \mathbf{Y}) \leq z^i (\alpha z^{-N} + (1 - \delta)) \mathbf{Y} \\
&\leq (1 - \frac{\delta}{2}) \mathbf{v}_i,
\end{aligned} \tag{3.6}$$

where $\mathbf{v}_i = (V(i, 1), V(i, 2), \dots, V(i, m))^t = z^i \mathbf{Y}$.

For any $0 < i \leq N$,

$$B_{-i} \mathbf{v}_0 + A_{-i+1} \mathbf{v}_1 + A_{-i+2} \mathbf{v}_2 + A_{-i+3} \mathbf{v}_3 + \cdots \leq \alpha \mathbf{e}^t + z^N A^*(z) \mathbf{Y} < \infty,$$

and for $i = 0$,

$$B_0 \mathbf{v}_0 + B_1 \mathbf{v}_1 + B_2 \mathbf{v}_2 + B_3 \mathbf{v}_3 + \cdots = B_0 \mathbf{Y} + B_+^*(z) \mathbf{Y} < \infty.$$

□

The equivalence of geometric ergodicity and the geometric stationary tail is interesting since results of ergodicity could directly lead to results of the stationary tail asymptotics and vice versa.

4 Strong ergodicity

In this section we show that the phase process A is not stochastic if and only if the $GI/G/1$ -type Markov chain is strongly ergodic.

Theorem 4.1 *The $GI/G/1$ -type Markov chain is strongly ergodic if and only if A is not stochastic.*

Proof: Assume first that A is not stochastic. Let $\tau_0 = \inf\{n \geq 0 : X_n \in L_0\}$, then by Proposition 3.3 on page 216 in Anderson [1], we only need to prove that there exists an $M < \infty$ such that

$$x_{i,r} := \mathbb{E}_{(i,r)} \tau_0 \leq M, \quad \text{for all } 1 \leq i < \infty \text{ and } 1 \leq r \leq m.$$

Let $\mathbf{X}_i = (x_{i,1}, \dots, x_{i,m})^t$ for $i \geq 1$ and $\mathbf{X}_0 = (x_{0,1}, \dots, x_{0,m_0})^t = \mathbf{0}$, then $\{\mathbf{X}_i : i \geq 0\}$ is the minimal non-negative solution of

$$\mathbf{Y}_i = \sum_{k \neq i} P_{i,k} \mathbf{Y}_k + \mathbf{e}^t, \quad i \geq 1, \tag{4.7}$$

where $\mathbf{Y}_0 = \mathbf{0}$.

Since A is irreducible and not stochastic, $(I - A)^{-1} = I + A + A^2 + \dots$ exists and is finite. In the following, we will prove that $\mathbf{X}_i \leq (I - A)^{-1}\mathbf{e}$. In fact, by a standard procedure for the minimal non-negative solution (for example, see [9]), set $\mathbf{X}_i^{(0)} = 0$, and for $n \geq 1$, let

$$\mathbf{X}_i^{(n)} = \sum_{k \neq i} P_{i,k} \mathbf{X}_k^{(n-1)} + \mathbf{e}^t, \quad i \geq 1, \quad \text{and } \mathbf{X}_0^{(n)} = 0,$$

we can then inductively prove that if $\mathbf{X}_i^{(n-1)} \leq (I - A)^{-1}\mathbf{e}^t$, then for $i \geq 1$,

$$\mathbf{X}_i^{(n)} \leq \sum_{k \neq i} P_{i,k} (I - A)^{-1}\mathbf{e}^t + \mathbf{e}^t \leq A(I - A)^{-1}\mathbf{e}^t + \mathbf{e}^t = (I - A)^{-1}\mathbf{e}^t.$$

Thus, $\mathbf{X}_i = \lim_{n \rightarrow \infty} \mathbf{X}_i^{(n)} \leq (I - A)^{-1}\mathbf{e}^t < \infty$.

For the converse, assume that A is stochastic. It is obvious that P is a Feller transition matrix, thus P cannot be strongly ergodic by Proposition 2.3 in Hou and Liu [10]. \square

5 Polynomial ergodicity

As indicated in the introduction, a sufficient condition for polynomial ergodicity was obtained in Højgaard and Møller [8]. In this section, we prove that it is also necessary. In addition, we provide another necessary and sufficient condition for polynomial ergodicity. To achieve this goal, in the first sub-section, we construct a control function h based on transition probabilities and provide a lower bound for the first hitting time. We also find a relationship between the first hitting times and transition probabilities. In the second sub-section, proofs of the main results are provided.

5.1 Lower bounds for the first hitting time

In this sub-section, we derive lower bounds for the first hitting time $\tau_i = \inf\{n \geq 1 : (X_n, Y_n) \in L_i\}$, for any finite i , which are useful to discuss necessary conditions for ergodicity. We provide a lemma for τ_0 . Similar results can be obtained for any $i \neq 0$ by the same argument. In fact, we can have a corresponding result for the first hitting time of any finite set instead of a level. We also obtain a relationship between the first hitting time and transition probabilities.

Lemma 5.1 *If $\sum_{k=0}^{\infty} k B_k \mathbf{e}^t < \infty^t$ and $\sum_{k=0}^{\infty} k A_k \mathbf{e}^t < \infty^t$, then (1) for i large enough, we have,*

$\mathbb{P}(C_i) > \frac{1}{2}$, where $C_i = \left\{ (X_s, Y_s) \in \left(\bigcup_{u=i}^{\lfloor i + \frac{i}{2} \rfloor} L_u \right) \text{ for } s = 0, 1, \dots, \left\lfloor \frac{i}{4\mu} \right\rfloor \right\}$ and $\lfloor x \rfloor$ denotes the largest integer equal to or smaller than x ; (2) for each sample on the event to C_i which satisfies (1),

$$\tau_0 - 1 \geq \frac{i - 1}{4\mu}, \tag{5.8}$$

where $\mu > 0$ is some fixed constant.

Proof: Since we consider the level independent $GI/G/1$ -type Markov chain, we have, for $i \geq 1$ and $j = 1, \dots, m$,

$$\mathbb{P}_{(i,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=i}^{i+(k-1)} L_u \right)^c \right) = \mathbb{P}_{(1,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right), \quad \text{for any } k > 0,$$

where L^c denotes the complement of L and $\mathbb{P}_{(i,j)}((X_1, Y_1) \in \cdot) = \mathbb{P}((X_1, Y_1) \in \cdot | (X_0, Y_0) = (i, j))$. Without loss of generality, we assume that

$$\begin{aligned} & \mathbb{P}_{(0,j_1)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right) \\ &= \max_j \left\{ \mathbb{P}_{(0,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right), \quad j = 1, \dots, m \right\}, \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}_{(1,j_2)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right) \\ &= \max_j \left\{ \mathbb{P}_{(1,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right), \quad j = 1, \dots, m \right\}. \end{aligned}$$

Let

$$h(k) = \max \left\{ \mathbb{P}_{(0,j_1)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right), \mathbb{P}_{(1,j_2)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right) \right\}.$$

Notice that

$$\sum_{k=1}^{\infty} \mathbb{P}_{(0,j_1)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right) < \infty,$$

and

$$\sum_{k=0}^{\infty} \mathbb{P}_{(1,j_2)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right) < \infty$$

since $\sum_{k=0}^{\infty} k B_k e^t < \infty^t$ and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$. Hence

$$\sum_{k=1}^{\infty} h(k) < \infty.$$

It is easy to know that for all $i \geq 0$, $j = 1, \dots, m$ and $k > 0$,

$$\mathbb{P}_{(i,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=i}^{i+(k-1)} L_u \right)^c \right) \leq h(k),$$

and $h(k)$ is a non-increasing function. Therefore there exists a sequence of *i.i.d.* random variables $W_n > 0$ with mean $\mu = \mathbb{E}[W_n]$ such that for all $i \geq 0$ and all $k \geq 0$,

$$\mathbb{P}_{(i,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=i}^{i+(k-1)} L_u \right)^c \right) \leq \mathbb{P}(W_n \geq k).$$

By the weak law of large numbers, we claim that for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(S_n \leq (\mu + \epsilon)n) = 1,$$

where $S_n = W_1 + \dots + W_n$. Hence there exists an N large enough such that for $n \geq N$,

$$\mathbb{P}(S_n < 2\mu n) \geq \frac{1}{2}.$$

Then, a stochastic comparison argument yields, for all $i \geq 0$ and all $n \geq N$,

$$\mathbb{P}_{(i,j)} \left((X_s, Y_s) \in \left(\bigcup_{u=i}^{\lfloor i+2\mu n \rfloor} L_u \right) \quad \text{for } s = 0, 1, \dots, n \right) \geq \frac{1}{2}. \quad (5.9)$$

For i large enough such that $\frac{i}{4\mu} \geq N$, we have from (5.9) with $n = \lfloor \frac{i}{4\mu} \rfloor \geq N$ that

$$\mathbb{P}_{(i,j)} \left((X_s, Y_s) \in \left(\bigcup_{u=i}^{\lfloor i+\frac{i}{2} \rfloor} L_u \right) \quad \text{for } s = 0, 1, \dots, \left\lfloor \frac{i}{4\mu} \right\rfloor \right) \geq \frac{1}{2}.$$

Set $C_i = \left\{ (X_s, Y_s) \in \left(\bigcup_{u=i}^{\lfloor i+\frac{i}{2} \rfloor} L_u \right) \quad \text{for } s = 0, 1, \dots, \left\lfloor \frac{i}{4\mu} \right\rfloor \right\}$, then we have that $\mathbb{P}(C_i) > \frac{1}{2}$. Therefore, for i large enough, we have, for each sample on the event C_i ,

$$\tau_0 - 1 \geq \left\lfloor \frac{i}{4\mu} \right\rfloor \geq \frac{i-1}{4\mu}.$$

□

Corollary 5.1 *Given any non-negative, non-decreasing and measurable function f , if $\mathbb{E}_{(0,j)}[f(\tau_0)] < \infty$ for $j = 1, 2, \dots, m$, $\sum_{k=0}^{\infty} k B_k e^t < \infty^t$, and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then for i large enough, we have*

$$\mathbb{E}_{(i,j)}[f(\tau_0 - 1)] \geq \frac{1}{2} f\left(\frac{i-1}{4\mu}\right), \quad (5.10)$$

where $\mu > 0$ is some fixed constant.

Proof: From Lemma 5.1, we know that for i large enough such that $\lfloor \frac{i}{4\mu} \rfloor \geq N$,

$$\mathbb{P}_{(i,j)} \left((X_s, Y_s) \in \left(\bigcup_{u=i}^{\lfloor i+\frac{i}{2} \rfloor} L_u \right) \quad \text{for } s = 0, 1, \dots, \left\lfloor \frac{i}{4\mu} \right\rfloor \right) \geq \frac{1}{2}. \quad (5.11)$$

Hence for i large enough, we have for the above event

$$\tau_0 - 1 \geq \left\lfloor \frac{i}{4\mu} \right\rfloor \geq \frac{i-1}{4\mu},$$

and therefore

$$f(\tau_0 - 1) \geq f\left(\left\lfloor \frac{i}{4\mu} \right\rfloor\right) \geq f\left(\frac{i-1}{4\mu}\right).$$

For i sufficiently large, this event has a probability of at least $\frac{1}{2}$ by (5.11) and therefore for i sufficiently large,

$$\mathbb{E}_{(i,j)}[f(\tau_0 - 1)] \geq \frac{1}{2}f\left(\frac{i-1}{4\mu}\right),$$

where $\mu = \mathbb{E}[W_n] > 0$ is some fixed constant. \square

Remark 5.1 Specifically, if for $l \in \{2, 3, \dots\}$, $f(x) = x^l$, then for i large enough, we have

$$\mathbb{E}_{(i,j)}[(\tau_0 - 1)^l] \geq \frac{1}{2}\left(\frac{i-1}{4\mu}\right)^l, \quad (5.12)$$

where $\mu > 0$ is some fixed constant and $\mathbb{E}_{(i,j)}[(\tau_0 - 1)^l] = \mathbb{E}[(\tau_0 - 1)^l | (X_0, Y_0) = (i, j)]$.

Next, we discuss the relationship between the first hitting time and one-step transition probabilities, which leads to a necessary condition for polynomial ergodicity.

Lemma 5.2 If for $l \in \{2, 3, \dots\}$, $\mathbb{E}_{(0,j)}[\tau_0^l] < \infty$, for $j = 1, \dots, m$, we have

$$\mathbb{E}_{(0,j)}[\tau_0^l] = \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)}[(\tau_0 + 1)^l] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)), \quad (5.13)$$

and

$$\mathbb{E}_{(1,j)}[\tau_0^l] = \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)}[(\tau_0 + 1)^l] \mathbb{P}_{(1,j)}((X_1, Y_1) = (k, i)), \quad (5.14)$$

where $\mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) = \mathbb{P}((X_1, Y_1) = (k, i) | (X_0, Y_0) = (0, j))$ and $\mathbb{P}_{(1,j)}((X_1, Y_1) = (k, i)) = \mathbb{P}((X_1, Y_1) = (k, i) | (X_0, Y_0) = (1, j))$.

Proof: Given $l \in \{2, 3, \dots\}$, we have that for $j = 1, \dots, m$,

$$\begin{aligned}
& \mathbb{E}_{(0,j)} \left[\tau_0^l \right] \\
&= \mathbb{E}_{(0,j)} \left[\mathbb{E} \left[\tau_0^l | (X_1, Y_1) \right] \right] \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E} \left[\tau_0^l | (X_1, Y_1) = (k, i) \right] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \sum_{n=1}^{\infty} n^l \mathbb{P}(\tau_0 = n | (X_1, Y_1) = (k, i)) \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \sum_{n=1}^{\infty} n^l \mathbb{P}(\tau_0 = n-1 | (X_0, Y_0) = (k, i)) \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \sum_{n=1}^{\infty} (n+1)^l \mathbb{P}(\tau_0 = n | (X_0, Y_0) = (k, i)) \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)} \left[(\tau_0 + 1)^l \right] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)).
\end{aligned}$$

Similarly, we can prove the other case. \square

Remark 5.2 In fact, for any non-negative and measurable function f , we can have decompositions corresponding to Lemma 5.2 for $f(\tau_0)$.

5.2 Necessary and sufficient conditions for polynomial ergodicity

In this sub-section, we discuss polynomial ergodicity for the $GI/G/1$ -type Markov chain and provide two necessary and sufficient conditions.

The $GI/G/1$ -type Markov chain is called polynomial ergodic of degree l for $l \in \{1, 2, \dots\}$, if for all $j = 1, \dots, m$,

$$\mathbb{E}_{(0,j)} \left[\tau_0^l \right] < \infty. \quad (5.15)$$

If $l = 1$ in equation 5.15, this definition coincides with the ordinary ergodicity.

We first use the lemmas from the previous sub-section to obtain a necessary condition for polynomial ergodicity.

Theorem 5.1 If for $l \in \{2, 3, \dots\}$, $\mathbb{E}_{(0,j)} \left[\tau_0^l \right] < \infty$ for all $j = 1, \dots, m$, and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then

$$\sum_k k^l A_k e^t < \infty^t,$$

and

$$\sum_k k^l B_k e^t < \infty^t.$$

Proof: Given $l \in \{2, 3, \dots\}$, by Corollary 5.1, for $j = 1, \dots, m$, we have for all $k > N$, where N is large enough,

$$\mathbb{E}_{(k,j)} \left[(\tau_0 - 1)^l \right] \geq \frac{1}{2} \left(\frac{k-1}{4\mu} \right)^l. \quad (5.16)$$

Then, by Lemma 5.2,

$$\begin{aligned} \mathbb{E}_{(0,j)} \left[\tau_0^l \right] &= \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)} \left[(\tau_0 + 1)^l \right] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\ &= \sum_{k=0}^N \sum_{i=1}^m \mathbb{E}_{(k,i)} \left[(\tau_0 + 1)^l \right] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\ &\quad + \sum_{k=N+1}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)} \left[(\tau_0 + 1)^l \right] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)). \\ &\geq \left[\sum_{k=N+1}^{\infty} k^l B_k \mathbf{e}^t \right]_j, \end{aligned}$$

where $\left[\sum_{k=N+1}^{\infty} k^l B_k \mathbf{e}^t \right]_j$ denotes the j th element of vector $\sum_{k=N+1}^{\infty} k^l B_k \mathbf{e}^t$. Hence

$$\sum_{k=N+1}^{\infty} k^l B_k \mathbf{e}^t < \infty^t,$$

and therefore

$$\sum_k k^l B_k \mathbf{e}^t < \infty^t.$$

Similarly, we can obtain

$$\sum_k k^l A_k \mathbf{e}^t < \infty^t.$$

□

Remark 5.3 *This theorem can be extended to a class of more general non-negative and non-decreasing rate functions.*

A necessary and sufficient condition for polynomial ergodicity can now be obtained since Højgaard and Møller [8] have already showed that the converse of Theorem 5.1 is also true. We state it as follows.

Theorem 5.2 *If $\sum_{k=0}^{\infty} k A_k \mathbf{e}^t < \infty^t$, then for $l \in \{2, 3, \dots\}$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$ for all $j = 1, \dots, m$ if and only if $\sum_k k^l A_k \mathbf{e}^t < \infty^t$ and $\sum_k k^l B_k \mathbf{e}^t < \infty^t$.*

Proof: For necessity, it follows from Theorem 5.3. \square

Next, we provide another necessary and sufficient condition for polynomial ergodicity based on the relationship between ergodicity and the tail behavior of the stationary distribution for the $GI/G/1$ -type Markov chain. The following lemma from Jarner and Tweedie [13] is needed.

Lemma 5.3 *Assume that for $l \in \{2, 3, \dots\}$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$ for $j = 1, \dots, m$, there exists a finite set (without loss of generality, we assume that this finite set is L_0) such that*

$$\sum_{i=0}^{\infty} \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} < \infty, \quad (5.17)$$

where π_{ij} is the j th element of π_i .

Using Lemma 5.2, Lemma 5.3 and factorization results, we have the following condition.

Theorem 5.3 *If $\sum_{k=0}^{\infty} k A_k e^t < \infty$, then for $l \in \{2, 3, \dots\}$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$ for all $j = 1, \dots, m$ if and only if $\sum_i i^l \pi_{ij} < \infty$.*

Proof: For necessity, by Corollary 5.1, for a given $l \in \{2, 3, \dots\}$ and all $j = 1, \dots, m$, we have for all $i > N$, where N is large enough,

$$\mathbb{E}_{(i,j)} [(\tau_0 - 1)^l] \geq \frac{1}{2} \left(\frac{i-1}{4\mu} \right)^l. \quad (5.18)$$

Then by Lemma 5.3,

$$\begin{aligned} \infty &> \sum_{i=0}^{\infty} \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} \\ &= \sum_{i=0}^N \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} + \sum_{i=N+1}^{\infty} \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} \\ &\geq \sum_{i=N+1}^{\infty} i^l \pi_{ij}. \end{aligned}$$

Thus

$$\sum_i i^l \pi_{ij} < \infty.$$

For sufficiency, we use factorization results and generating function techniques in our analysis. From (2.3) we have that for $0 < z < 1$,

$$\pi^*(z)[I - R^*(z)] = \pi_0 R_0^*(z),$$

and thus

$$\pi^*(z) = \pi_0^* R_0^*(z)[I - R^*(z)]^{-1}, \quad (5.19)$$

since $I - R(z)$ is invertible. Taking the l th ($l = 2, 3, \dots$) derivative on the both sides of equation (5.19), we have

$$\begin{aligned}
& \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) z^{k-l} \pi_k \\
&= \pi_0 \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) z^{k-l} R_{0,k} \sum_{k=1}^{\infty} z^k \sum_{n=0}^{\infty} R_k^{n*} \\
& \quad + \pi_0 \sum_{k=1}^{\infty} z^k R_{0,k} \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) z^{k-l} \sum_{n=0}^{\infty} R_k^{n*} + \mathbf{c}_{l-1}(z),
\end{aligned} \tag{5.20}$$

where $\mathbf{c}_{l-1}(z)$ is the summation of all terms of the form

$$\pi_0 \sum_{k=p-1}^{\infty} k(k-1) \cdots (k-p+1) z^{k-p} R_{0,k} \sum_{k=q-1}^{\infty} k(k-1) \cdots (k-q+1) z^{k-q} \sum_{n=0}^{\infty} R_k^{n*}$$

and

$$p + q \leq l - 1.$$

There are only finitely many such terms of this kind. Let $z \rightarrow 1-$ in (5.20), we obtain

$$\begin{aligned}
& \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \pi_k \\
&= \pi_0 \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) R_{0,k} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} R_k^{n*} \\
& \quad + \pi_0 \sum_{k=1}^{\infty} R_{0,k} \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \sum_{n=0}^{\infty} R_k^{n*} + \mathbf{c}_{l-1}(1).
\end{aligned}$$

Since $\sum_i i^l \pi_i < \infty$ and

$$\lim_{k \rightarrow \infty} \frac{k(k-1) \cdots (k-l+1)}{k^l} = 1,$$

it follows that

$$\sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \pi_k < \infty.$$

Therefore

$$\sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) R_{0,k} < \infty$$

and

$$\sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \sum_{n=0}^{\infty} R_k^{n*} < \infty.$$

Thus

$$\sum_{k=1}^{\infty} k^l R_{0,k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^l \sum_{n=0}^{\infty} R_k^{n*} < \infty.$$

From Theorem 1 in [7] and Theorem 12 in [24], we have for $k \geq 1$, $R_{0,k} \geq B_k$ and $R_k \geq A_k$. Hence,

$$\sum_{k=1}^{\infty} k^l B_k e^t \leq \sum_{k=1}^{\infty} k^l R_{0,k} e^t < \infty^t$$

and

$$\sum_{k=1}^{\infty} k^l A_k e^t \leq \sum_{k=1}^{\infty} k^l \sum_{n=0}^{\infty} A_k^{n*} e^t \leq \sum_{k=1}^{\infty} k^l \sum_{n=0}^{\infty} R_k^{n*} e^t < \infty^t.$$

Finally, by Theorem 5.2, we know that $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$. □

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